Equations of elasticity: Cartesian grid

Equations

In this model **u** = (U, V, W) is vector of displacement.
The equations of X-force balance are:

$$0=\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+f_x \tag{1}$$

The displacements and the stresses are connected by Hooke's law:

$$\begin{gathered}\sigma_{xx}=2G\varepsilon_x+LD-HT,\\ \sigma_{yy}=2G\varepsilon_y+LD-HT,\\ \sigma_{zz}=2G\varepsilon_z+LD-HT,\\ \sigma_{xy}=G\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right),\ \sigma_{xz}=G\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right),\ \sigma_{yz}=G\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right),\\ D=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz},\\ \varepsilon_{xx}=\frac{\partial U}{\partial x},\ \varepsilon_{yy}=\frac{\partial V}{\partial y},\ \varepsilon_{zz}=\frac{\partial W}{\partial z}\end{gathered} \tag{2}$$

here

D - dilatation (relative changing of volume at deformation),
G,L - first (μ) and second (λ) Lame's constant:
E - Young's module,
P - Poisson's coefficient,
α - thermal-expansion coefficient,

$$G=\frac{E}{2(1+P)},L=\frac{EP}{(1+P)(1-2P)}.$$

$$H=\frac{\alpha E}{1-2P}$$

Use (2) can be written (1) as

$$\begin{gathered}\left\{\frac{\partial}{\partial x}\left[(2G+L)\frac{\partial U}{\partial x}\right]+\frac{\partial}{\partial y}\left[G\frac{\partial U}{\partial y}\right]+\frac{\partial}{\partial z}\left[G\frac{\partial U}{\partial z}\right]\right\}+f_x+\\ \frac{\partial}{\partial x}\left[L(\varepsilon_{yy}+\varepsilon_{zz})-HT\right]+\ \frac{\partial}{\partial y}\left[G\frac{\partial V}{\partial x}\right]+\frac{\partial}{\partial z}\left[G\frac{\partial W}{\partial x}\right]=0\end{gathered} \tag{3}$$

The Equations for u_y and u_z have a similar type.
The terms in figured bracket {} is standard div(grad) term Phoenics with anisotropy diffusion.

For **two-dimensional** (x,y) problems, the U-equation simplifies to:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+f_x=0,$$

$$\sigma_{xx}=(2G+L)\left(\frac{\partial U}{\partial x}\right)+L\varepsilon_{yy}+L\varepsilon_{zz}-H\varepsilon_t,$$

$$\sigma_{xy}=G\left(\frac{\partial U}{\partial y}\right)+G\frac{\partial V}{\partial x}$$

Stresses and strains in the z-direction are linked by the equation:

$$\sigma_{zz}=-b_z\varepsilon_{zz} \tag{3a}$$

where b_z is a parameter which expresses the extent to which the material is free to expand in that direction, which may vary between zero (for free expansion) and infinity (for absolute constraint).

With (3a) can be written as

$$\varepsilon_{zz} = \frac{1}{L + 2G + b_z}\left[-L\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + H\varepsilon_t\right]$$

For what is known as the *Plane-Strain* model,

$$b_z = \infty,\ \varepsilon_{zz} = 0, \sigma_{zz} = L(\varepsilon_{xx} + \varepsilon_{yy}) - H\varepsilon_t,$$

while for what is known as the *Plane-Stress* model,

$$b_z = 0,\ \sigma_{zz} = 0,\ \ \varepsilon_{zz} = \frac{1}{L + 2G}\left[-L\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + H\varepsilon_t\right]$$

For one-dimensional (x) problems

$$\sigma_{zz} = -b_z\varepsilon_{zz},\ \sigma_{yy} = -b_y\varepsilon_{yy}$$

and

$$\varepsilon_{zz} = \frac{1}{L + 2G + b_z}\frac{1 - d_y}{1 - d_y d_z}\left[-L\varepsilon_{xx} + H\varepsilon_t\right],$$

$$\varepsilon_{yy} = \frac{1}{L + 2G + b_y}\frac{1 - d_z}{1 - d_y d_z}\left[-L\varepsilon_{xx} + H\varepsilon_t\right],$$

$$d_y = \frac{L}{L + 2G + b_y},\ d_z = \frac{L}{L + 2G + b_z}$$

FVE

We shall use standard approximation for fluxes:

$$\begin{aligned}
&\left\{\left(2G + L\right)\frac{\partial U}{\partial x}\right\}_E \Delta A_E - \left\{\left(2G + L\right)\frac{\partial U}{\partial x}\right\}_P \Delta A_P + \\
&+\left\{G\frac{\partial U}{\partial y}\right\}_{ne} \Delta A_{en} - \left\{G\frac{\partial U}{\partial y}\right\}_{se} \Delta A_{es} \\
&+\left\{G\frac{\partial U}{\partial z}\right\}_{he} \Delta A_{eh} - \left\{G\frac{\partial U}{\partial Z}\right\}_{le} \Delta A_{el} + S_{int,e} + f_{x,e}\Delta V_e = 0
\end{aligned} \tag{4}$$

where

$$\begin{aligned}S_{int,e} = &\left\{L_E\left(\varepsilon_{yy,E}+\varepsilon_{zz,E}\right)\Delta A_E - L_P\left(\varepsilon_{yy,P}+\varepsilon_{zz,P}\right)\Delta A_P\right\}+\\ &-\left\{H_E T_E \Delta A_E - H_P T_P \Delta A_P\right\}+\\ &\left\{\left(\frac{\partial V}{\partial x}\right)_{en}\frac{\Delta A_n G_n+\Delta A_{En}G_{En}}{2}-\left(\frac{\partial V}{\partial x}\right)_{es}\frac{\Delta A_s G_s+\Delta A_{Es}G_{Es}}{2}\right\}+\\ &\left\{\left(\frac{\partial W}{\partial x}\right)_{eh}\frac{\Delta A_h G_h+\Delta A_{Eh}G_{Eh}}{2}-\left(\frac{\partial W}{\partial x}\right)_{el}\frac{\Delta A_l G_l+\Delta A_{El}G_{El}}{2}\right\},\end{aligned} \qquad (5)$$

$$\left(\frac{\partial V}{\partial x}\right)_{en}=\frac{\left(V_{En}-V_n\right)}{\delta x_e},\ \left(\frac{\partial V}{\partial x}\right)_{es}=\frac{\left(V_{Es}-V_s\right)}{\delta x_e},\ldots\ldots$$

$$G_n=\frac{\Delta y_P+\Delta y_N}{\dfrac{\Delta y_P}{G_P}+\dfrac{\Delta y_N}{G_N}},\ \ldots..$$

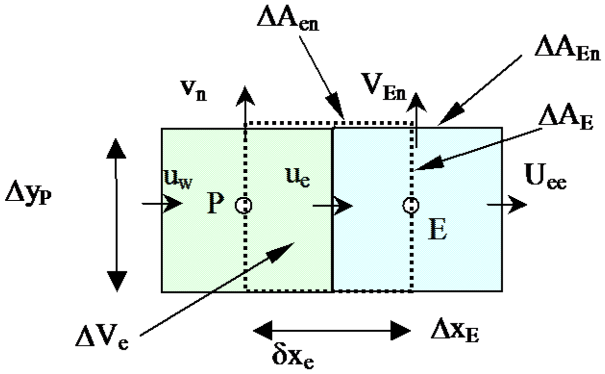


Fig. 1 Internal u_x-Cell

<u>Boundary Condition</u>

<u>Tangential boundary</u> (en, Fig. 1) :

$$G\frac{\partial u_x}{\partial y}=\tau_0-G\frac{\partial u_y}{\partial x}$$

At a FVE-level possible to unite two members:

$$+\left\{G\frac{\partial U}{\partial y}\right\}_{ne}\Delta A_{en}$$

and

$$\left(\frac{\partial V}{\partial x}\right)_{en}\frac{\Delta A_n G_n+\Delta A_{En}G_{En}}{2}$$

Summary we shall get on boundary

$$+\left\{G\frac{\partial U}{\partial y}\right\}_{ne}\Delta A_{en}+\left(\frac{\partial V}{\partial x}\right)_{en}\frac{\Delta A_n G_n+\Delta A_{En}G_{En}}{2}=$$
$$\left\{G\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right\}_{ne}\Delta A_{en}=\tau_{0,ne}\Delta A_{en} \qquad (6)$$

I.e. on boundary by means of patch it is necessary to assign tangential stresses itself!

Normal boundary (e, Fig. 2) :

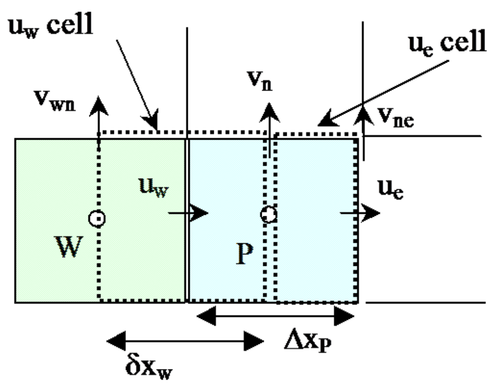


Fig. 2 Boundary u_x-Cell

Boundary condition :

$$(2G+L)\left(\frac{\partial u_x}{\partial x}\right)=\sigma_{xx}-L\varepsilon_y-L\varepsilon_z+HT$$

Uniting members FVE, shall get

$$\left\{(2G+L)\frac{\partial U}{\partial x}\right\}_E\Delta A_E+L_E\left(\varepsilon_{yy,E}+\varepsilon_{zz,E}\right)\Delta A_E-H_E T_E\Delta A_E=\sigma_{0,e}\Delta A_E \qquad (7)$$

I.e. on boundary by means of patch it is necessary to assign normal stresses itself!

Equations of elasticity: Polar grid

Initial equations - a Hooke's law ($U = u_\theta$, $V = u_r$, $W = u_z$):

$$
\begin{aligned}
&\sigma_{rr} = 2G\varepsilon_{rr} + Le - HT, \quad e_{rr} = \frac{\partial V}{\partial r}, \\
&\sigma_{\theta\theta} = 2G\varepsilon_{\theta\theta} + Le - HT, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{V}{r}, \\
&\sigma_{zz} = 2G\varepsilon_{zz} + Le - HT, \quad e_{zz} = \frac{\partial W}{\partial z}, \\
&\sigma_{r\theta} = G\left(\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial U}{\partial r} - \frac{U}{r}\right), \\
&\sigma_{rz} = G\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r}\right), \\
&\sigma_{z\theta} = G\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{\partial U}{\partial z}\right), \\
&e = e_{rr} + e_{\theta\theta} + e_{zz}
\end{aligned}
\tag{8}
$$

and balance of force equation

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + f_r = 0, \\
&\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{\theta r}) + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{r\theta}}{r} + f_\theta = 0, \\
&\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0,
\end{aligned}
\tag{9}
$$

With (4.1) and (4.2) can be V-equations written as

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r(2G+L)\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(G\frac{1}{r}\frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(G\frac{\partial V}{\partial z}\right) + \\
&\frac{1}{r}\frac{\partial}{\partial r}r\left(L(e_{\theta\theta}+e_{zz}) - HT\right) + \frac{1}{r}\frac{\partial}{\partial \theta}G\left(\frac{\partial U}{\partial r} - \frac{U}{r}\right) + \frac{\partial}{\partial z}\left(G\frac{\partial W}{\partial r}\right) - \\
&-\frac{1}{r}\left[2G\left(\frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{V}{r}\right) + L\varepsilon_{rr} + L\left(e_{\theta\theta}+e_{zz}\right) - HT\right] + f_r = 0,
\end{aligned}
$$

OR

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r(2G+L)\frac{\partial V}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(G\frac{1}{r}\frac{\partial V}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(G\frac{\partial V}{\partial z}\right)\right\}+$$

$$\frac{\partial}{\partial r}\left(L(e_{\theta\theta}+e_{zz})-HT\right)+\frac{1}{r}\frac{\partial}{\partial \theta}G\left(\frac{\partial U}{\partial r}-\frac{U}{r}\right)+\frac{\partial}{\partial z}\left(G\frac{\partial W}{\partial r}\right)- \quad (10)$$

$$-\frac{1}{r}\left[2G\frac{1}{r}\frac{\partial U}{\partial \theta}+L\varepsilon_{rr}\right]-2G\frac{V}{r^2}+f_r=0,$$

This is polar analog main Cartesian equation (3). Similar equations possible to write for U and W component:

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rG\frac{\partial U}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left((2G+L)\frac{1}{r}\frac{\partial U}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(G\frac{\partial U}{\partial z}\right)\right\}+$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(L(e_{rr}+e_{zz})-HT+(2G+L)\frac{V}{r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left[rG\left(\frac{1}{r}\frac{\partial V}{\partial \theta}-\frac{U}{r}\right)\right]+\frac{\partial}{\partial z}\left(G\frac{1}{r}\frac{\partial W}{\partial \theta}\right)+$$

$$+G\frac{1}{r}\left(\frac{1}{r}\frac{\partial V}{\partial \theta}+\frac{\partial U}{\partial r}-\frac{U}{r}\right)+f_\theta=0,$$

(11)

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rG\frac{\partial W}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(G\frac{1}{r}\frac{\partial W}{\partial \theta}\right)+\frac{\partial}{\partial z}\left((2G+L)\frac{\partial W}{\partial z}\right)\right\}+$$

$$\frac{\partial}{\partial z}\left(L(e_{rr}+e_{\theta\theta})-HT\right)+\frac{1}{r}\frac{\partial}{\partial r}\left[rG\frac{\partial V}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left(G\frac{\partial U}{\partial z}\right)+f_z=0, \quad (12)$$

<u>Boundary Condition on FGE-level</u>

<u>Displacement U (u_θ) – tangential boundary (N/S and H/L).</u>
Similar doing we have, either as in Cartesian case - two members unite on boundaries:

$$\left\{G\frac{\partial U}{\partial r}\right\}_{ne}\Delta A_{en}+\left[\left(\frac{\partial V}{\partial \theta}\right)_{en}-U_{en}\right]\frac{\Delta A_n G_n+\Delta A_{En}G_{En}}{2r_n}-$$

$$\left\{G\left(\frac{\partial U}{\partial r}+\frac{1}{r}\frac{\partial V}{\partial \theta}-\frac{U}{r}\right)\right\}_{ne}\Delta A_{en}=\tau_{0,ne}\Delta A_{en}$$

<u>Displacement U (u_θ) – normal boundary (E/W).</u>
Uniting members FVE, shall get

$$\left\{(2G+L)\left(\frac{1}{r}\frac{\partial U}{\partial \theta}+\frac{V}{r}\right)\right\}_E\Delta A_E+L_E\left(\varepsilon_{rr,E}+\varepsilon_{zz,E}\right)\Delta A_E-H_E T_E\Delta A_E=\sigma_{0,e}\Delta A_E$$

<u>Displacement V (u_r) – tangential boundary (E/W and H/L).</u>
Similar doing

$$\left\{G\frac{1}{r}\frac{\partial V}{\partial \theta}\right\}_{tan}\Delta A_{tan}+\left[\left(\frac{\partial U}{\partial r}\right)-U\right]_{tan}\frac{\Delta A_e G_e+\Delta A_{Ne}G_{Ne}}{2r_e}-$$

$$\left\{G\left(\frac{\partial U}{\partial r}+\frac{1}{r}\frac{\partial V}{\partial \theta}-\frac{U}{r}\right)\right\}_{tan}\Delta A_{tan}=\tau_{0,tan}\Delta A_{tan}$$

<u>Displacement V (u_r) – normal boundary (N/S).</u>
Here appears the ***problem*** with association of the members. This is connected with difference by form of the members in V-equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r(2G+L)\frac{\partial V}{\partial r}\right)\quad\text{and}\quad\frac{\partial}{\partial r}\left(L(e_{\theta\theta}+e_{zz})-HT\right)$$

On FGE-level

$$\left\{(2G+L)\frac{\partial V}{\partial y}\right\}_N\Delta A_N-\left\{(2G+L)\frac{\partial V}{\partial y}\right\}_P\Delta A_P\quad\text{and}$$

$$\frac{\Delta A_N+\Delta A_P}{2}\left\{\left[L_N\left(\varepsilon_{\theta\theta,N}+\varepsilon_{zz,N}\right)-H_N T_N\right]-\left[L_P\left(\varepsilon_{\theta\theta,P}+\varepsilon_{zz,P}\right)-H_P T_P\right]\Delta A_P\right\}$$

On N-boundary can to write

$$\left\{(2G+L)\frac{\partial V}{\partial y}\right\}_n\Delta A_n+\frac{\Delta A_n+\Delta A_P}{2}\left\{\left[L_n\left(\varepsilon_{\theta\theta,n}+\varepsilon_{zz,n}\right)-H_n T_n\right]\right\}=$$

$$\left\{(2G+L)\frac{\partial V}{\partial y}+L\left(\varepsilon_{\theta\theta}+\varepsilon_{zz}\right)-HT\right\}_n\Delta A_n+\frac{\Delta A_P-\Delta A_n}{2}\left\{\left[L_n\left(\varepsilon_{\theta\theta,n}+\varepsilon_{zz,n}\right)-H_n T_n\right]\right\}=$$

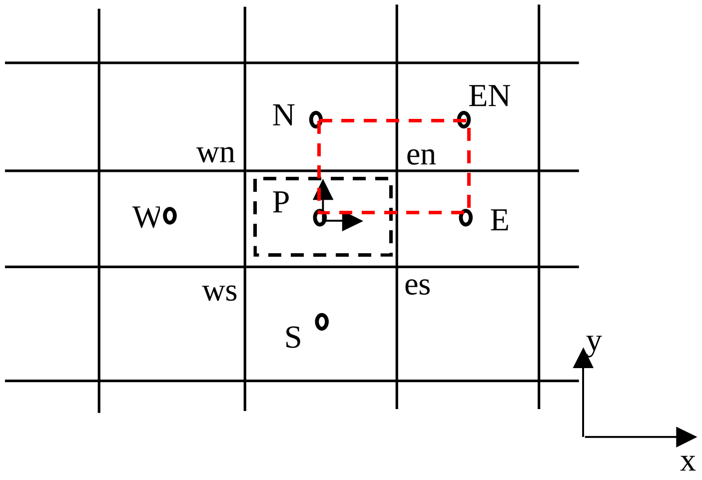
$$\sigma_{0,n}\Delta A_n+\frac{\Delta A_P-\Delta A_n}{2}\left\{\left[L_n\left(\varepsilon_{\theta\theta,n}+\varepsilon_{zz,n}\right)-H_n T_n\right]\right\}$$

<u>Displacement W (u_z) – tangential and normal boundary.</u>
Similarly Cartesian coordinate system

Add VIA 12.03.2007

Colocate Displacements Mehod

We shall consider FVE for 2D(x,y) cartesian grid when displacements are determined in the centre scalar cell.



The equations of X-force balance are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0, \tag{13}$$

The displacements and the stresses are connected by Hooke's law:

$$\begin{aligned}
&\sigma_{xx} = (2G + L_m)\varepsilon_x + L_m \varepsilon_y - H_m T,\\
&\sigma_{yy} = (2G + L_m)\varepsilon_y + L_m \varepsilon_x - H_m T,\\
&\sigma_{zz} = -b_z \varepsilon_z,\\
&\sigma_{xy} = G\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right),\\
&\varepsilon_{xx} = \frac{\partial U}{\partial x},\ \varepsilon_{yy} = \frac{\partial V}{\partial y},\\
&\varepsilon_{zz} = -\frac{1}{L + 2G + b_z}\left[L\left(\varepsilon_x + \varepsilon_y\right) - HT\right],\\
&L_m = L\frac{2G + b_z}{2G + b_z + L}, H_m = H\frac{2G + b_z}{2G + b_z + L}
\end{aligned} \tag{14}$$

Integrating equation (13) on scalar cell P, possible get FVE

$$\left\{(2G + L_m)\frac{\partial U}{\partial x}\right\}_e \Delta A_e - \left\{(2G + L_m)\frac{\partial U}{\partial x}\right\}_w \Delta A_w + \\ + \left\{G\frac{\partial U}{\partial y}\right\}_n \Delta A_n - \left\{G\frac{\partial U}{\partial y}\right\}_s \Delta A_s + S_{int,P} + f_{x,P}\Delta V_P = 0 \tag{15}$$

Where internal source consists of gradient and tangential terms:

$$S_{int,P} = \left\{ \left[L_{m,e} \frac{V_{en} - V_{es}}{\Delta y_P} - H_{m,e} T_e \right] \Delta A_e - \left[L_{m,w} \frac{V_{wn} - V_{ws}}{\Delta y_P} - H_{m,w} T_w \right] \Delta A_w \right\} + \left\{ G_n \frac{V_{en} - V_{wn}}{\Delta x_P} \Delta A_n - G_s \frac{V_{es} - V_{ws}}{\Delta x_P} \Delta A_s \right\} \tag{16}$$

For calculation all property-coefficients on faces of cells is used harmonic interpolation, for instance

$$\frac{\Delta y_P + \Delta y_N}{G_n} = \frac{\Delta y_P}{G_P} + \frac{\Delta y_N}{G_N},$$

For calculation of the temperature on faces is used linear interpolation

$$T_e = T_P + \frac{T_E - T_P}{\delta x_e} \frac{\Delta x_P}{2},$$

For calculation of displacements in corners (the centre edges in z-direction), for instance Ven shall consider "red"-rectangle P-E-E-N. We Shall consider that in this rectangle

$$V = V_P + \left(\frac{\partial V}{\partial x} \right)_P (x - x_P) + \left(\frac{\partial V}{\partial y} \right)_P (y - y_P) + \left(\frac{\partial^2 V}{\partial x \partial y} \right)_P (x - x_P)(y - y_P)$$

and

$$\left(\frac{\partial V}{\partial x} \right)_P = \frac{V_E - V_P}{\delta x_e}, \left(\frac{\partial V}{\partial y} \right)_P = \frac{V_N - V_P}{\delta y_n},$$

$$\left(\frac{\partial^2 V}{\partial x \partial y} \right)_P = \frac{\dfrac{V_N - V_{EN}}{\delta x_e} - \left(\dfrac{\partial V}{\partial x} \right)_P}{\delta y_n}$$

Boundary Condition

The Boundary conditions are assigned similarly mate the staggered displacements!